# On Kähler manifolds with positive orthogonal bisectional curvature

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# 1 Introduction

The famous Frankel conjecture asserts that any compact Kähler manifold with positive bisectional curvature must be biholomorphic to $\mathbb{CP}^n$. This conjecture was settled affirmatively in early 1980s by two groups of mathematicians independently: Siu-Yau[16] via differential geometry method and Morri [15] by algebraic method. There are many interesting papers following this celebrated work; in particular to understand the classification of Kähler manifold with non-negative bisectional curvature, readers are referred to N. Mok's work [14] for further references. In 1982, R. Hamilton [10] introduced the Ricci flow as a means to deform any Riemannian metric in a canonical way to a Einstein metric. He particularly showed that, in any 3-dimensional compact manifold, the positive Ricci curvature is preserved by the Ricci flow. Moreover, the Ricci flow deforms the metric more and more towards Einstein metric. Consequently, he proved that the underlying manifold must be diffeomorphic to $S^3$ or a finite quotient of $S^3$. By a theorem of M. Berger in the 1960s, any Kähler Einstein metric with positive bisectional curvature is the Fubni-Study metric (with constant bisectional curvature). A natural and long standing problem for Kähler Ricci flow is: In $\mathbb{CP}^n$, is Kähler Ricci flow converges to the Fubni-Study metric if the initial metric has positive bisectional curvature? There are many interesting work in this direction in 1990s (c.f. [1] [14] ) and the problem was completely settled in 2000 by [5] [6] affirmatively. One key idea is the introduction of a series new geometrical functionals which play a crucial role in deriving the bound of scalar curvature, diameter and Sobleve constants etc..

It is well known that the positivity of bisectional curvature is preserved along the Kähler Ricci flow, due to S. Bando [1] in dimension 3 and N. Mok in general dimension [14]. Following the work of N. Mok, in an unpublished work of Cao-Hamilton, they claimed that the positive orthogonal bisectional curvature is also preserved under the Kähler Ricci flow.

In any Kähler manifold, we can split the space of $(1,1)$ forms into two orthogonal components: the line spanned by the Kähler form, and its orthogonal complement $\Lambda_0^{1,1}$. The traceless part of the bisectional curvature can be viewed as an operator acting on this subspace $\Lambda_0^{1,1}$. We call a Kähler metric has 2-positive traceless bisectional curvature if the sum of any two eigenvalues of the tracless bisectional curvature operator in $\Lambda_0^{1,1}$ is positive. In complex dimension 2, this property was studied by [12] where they proved that the 2-positive traceless bisectional curvature condition is preserved under the Kähler Ricci flow. This result, as well as the method of proof, is similar to that of H. Chen in Riemannian case. In [8], H. Li and the author prove that the 2-positive traceless bisectional curvature operator is preserved under the Kähler Ricci flow in all dimensions. Moreover, we show that in [8] that Kähler metric with 2-positive traceless bisectional curvature must also have positive orthogonal bisectional curvature. Therefore, if the initial Kähler metric has 2-positive traceless bisectional curvature, then the orthogonal bisectional curvature remains positive on

the entire Kähler Ricci flow.

In this paper, we study any Kähler manifold where the positive orthogonal bisectional curvature is preserved on the Kähler Ricci flow. Naturally, we always assume that the first Chern class $C_1$ is positive. Under this assumption, we first show that various inequalities (convex cone), on the scalar curvature, the Ricci curvature tensor and the holomorphic sectional curvature tensor, are preserved over the Kähler Ricci flow respectively. We then discuss geometrical application of these results. In particular, we prove that any irreducible Kähler manifold with positive orthogonal bisectional curvature must be biholomorphic to $\mathbb{CP}^n$. This can be viewed as a generalization of Siu-Yau[16], Morri's solution [15] of the Frankel conjecture.

Now we state some results on maximum principle first.

**Theorem 1.1.** *Along the Kähler Ricci flow, the following statement hold*

1. *The lower bound of the scalar curvature, if $\leq 0$, is preserved and improved over time. The lower bound will increase to* $0$ *exponentially.*

2. *The upper bound of the scalar curvature grows at most exponentially.*

**Theorem 1.2.** *If the orthogonal bisectional curvature is positive along the Kähler Ricci flow, then the following hold:*

1. *If the initial metric has positive Ricci curvature, then this condition will be preserved.*

2. *If the initial Ricci curvature is not positive, then the lower bound of the Ricci curvature will increase. As $t \to \infty$, this lower bound will at least increase to* $0$ *as $t \to \infty$.*

**Remark 1.3.** *The second part of Theorem 1.1.1 was proved firs by B. Chow [9] in $S^2$. Statements in Theorem 1.1 should be known to the experts in the field and the proof in high dimension is similar to 2-d case.*

*Theorem 1.2.1 was observed by Cao-Hamilton first.*

**Theorem 1.4.** *If the positivity of the orthogonal bisectional curvature is preserved along the Kähler Ricci flow, then the lower bound of the holomorphic sectional curvature, if non-positive, is preserved under the flow. Moreover, the lower bound will be improved to* $0$ *exponentially over the flow.*

The following theorem is more or less technical.

**Theorem 1.5.** *On the Kähler manifold where the positivity of the orthogonal bisectional curvature is preserved under the Kähler Ricci flow, suppose that the condition $Ric \geq \nu > 0$ is preserved over the entire flow. Then the lower bound of the holomorphic sectional curvature, if $\leq 0$, will become positive after finite time. Moreover, if the lower bound of bisectional curvature is positive and $\nu > \frac{1}{2}$, then this lower bound will be increased over time and approach $\frac{2\nu-1}{n+1}$ exponentially.*

The following question is then very natural

**Question/Conjecture 1.6.** *Is an irreducible compact Kähler manifold with positive orthogonal bisectional curvature necessary $\mathbb{CP}^n$?*

A related question is: for higher dimension, does manifold with positive orthogonal bisectional curvature necessary has first Chern class positive? Apparently, we need to assume $n > 1$. Hopefully, when $n$ is large enough, the positive orthogonal bisectional curvature condition implies the positivity of first Chern class?

Following [6], we have

**Theorem 1.7.** *On a Kähler Einstein manifold where the positive orthogonal bisectional curvature is preserved over the Kähler Ricci flow, the flow converges to a Kähler Einstein metric exponentially fast.*

Comparing to Theorem 1.1 and Remark 1.9 of [6], we drop the assumption that the initial metric has positive Ricci curvature. Theorem 1.2 implies that, after finite time, the Ricci curvature will be bigger than $-1$. The energy functional $E_1$ will be monotonely decreased afterwards. The rest of the proof is the same as in [6]. Here $E_1$ is one of a set of new geometrical functional introduced in [6].

Invoking a deep theorem of Perelman where he shows that the scalar curvature is uniformly bounded along the Kähler Ricci flow with any smooth initial metric[1], then we prove

**Theorem 1.8.** *For any irreducible Kähler manifold which admits positive orthogonal bisectional curvature and $C_1 > 0$, if this positivity condition is preserved under the flow, then the underlying manifold is biholomorphic to $\mathbb{CP}^n$.*

As a corollary, we have

**Theorem 1.9.** *Any irreducible Kähler manifold with positive orthogonal bisectional curvature or positive 2-traceless bisectional curvature and $C_1 > 0$ must be $\mathbb{CP}^n$.*

Following proof of Theorem 1.8 and classification of manifold with non-negative bisectional curvature (c.f. [14]), one shall be able to generalize Theorem 1.8 to the case of non-negative orthogonal bisectional curvature positive case.

One unsatisfactory feature of the present proof is that we had to take limit as $t \to \infty$ first in order to show that the bisectional curvature becomes positive after finite time. The argument is a little bit in-direct. If a more direct argument can be obtained, then perhaps one can avoid using of Perelman's theorem on the scalar curvature function. The following is a weak, but direct theorem.

[1]Using Perelman's local estimates together with Cao's Harnack inequalities under this case, Cao-Zhu-Zhu [3] gave another proof of a uniform bound of the scalar curvature under the assumption of positive bisectional curvature.

**Theorem 1.10.** *For any Kähler manifold with positive orthogonal bisectional curvature and $C_1 > 0$ , then the following statements are equivalent*

1. *there exists a lower bound of the energy functional $E_1$.*
2. *there exists a lower bound of the Mabuchi energy;*
3. *there is a Kähler Einstein metric in the Kähler class;*

*In particular, the flow converges exponentially fast to the Fubni-Study metric and the underlying manifold is $\mathbb{CP}^n$.*

Note that in the proof of this theorem, we don't use Frankel Conjecture.

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# 2 The Maximum principle along the Kähler Ricci flow

In this section, we prove some theorems on the scalar curvature, Ricci curvature and holomorphic sectional curvature via Hamilton's maximum principle on tensors along a geometric flow.

## 2.1 On the scalar curvature

Here we give a proof to Theorem 1.1.

*Proof.* The evolution equation for Kähler potential is:

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi.$$

Set

$$F(t) = |\nabla \frac{\partial \varphi}{\partial t}\ |^2_{\varphi(t)}\ .$$

Then the evolution equation for $F(t)$ is simple

$$\frac{\partial}{\partial t} F = \triangle_{\varphi(t)} F + F - \left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\bar{\beta}} \cdot \left(\frac{\partial \varphi}{\partial t}\right)_{\bar{\alpha}\beta} - \left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\beta} \cdot \left(\frac{\partial \varphi}{\partial t}\right)_{\bar{\alpha}\bar{\beta}}.$$

Note that

$$\left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} - R_{\alpha\bar{\beta}}.$$

Recalled the evolution equation for the scalar curvature

$$\begin{array}{rcl} \frac{\partial R}{\partial t} & = & \triangle R + |Ric - 1|^2 + 2R - n - R \\ & = & \triangle R + \left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\bar{\beta}} \cdot \left(\frac{\partial \varphi}{\partial t}\right)_{\bar{\alpha}\beta} + R - n. \end{array}$$

Set $h = R - n + F$. Then the evolution equation for $h$ is

$$\frac{\partial h}{\partial t} \quad = \quad \triangle h + h - \left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\beta} \cdot \left(\frac{\partial \varphi}{\partial t}\right)_{\bar{\alpha}\bar{\beta}}.$$

Thus, the upper bound of $h$ grows at most exponentially

$$h \le \max_{x \in M} \ h(x, 0) e^t.$$

Consequently, we have

$$R(x, t) \le C_1 e^t + n, \qquad \forall t > 0.$$

To obtain lower bound, let $-\mu(t)$ be the negative lower bound of the scalar curvature at time $t$. Then

$$\begin{array}{rcl} \frac{\partial}{\partial t}(R + \mu(t)) & = & \triangle(R + \mu(t)) + Ric^2 - (R + \mu(t)) + \mu(t) + \mu(t)' \\ & \ge & \triangle(R + \mu(t)) - (R + \mu(t)) \end{array}$$

where we set

$$\mu'(t) + \mu(t) = 0, \qquad \mu(0) > -\min_{x \in M} R(x, 0) > 0.$$

In particular, if $\min_{x \in M} \ R(x, 0) < 0$, then

$$R(x, t) \ge \min_{x \in M} R(x, 0) \ e^{-t}.$$

□

## 2.2 On the Ricci curvature

Now we give a proof of Theorem 1.2.

*Proof.* Suppose $-\mu(t) < 0$ is lower bound of Ricci curvature (not necessary optimal). Set

$$\hat{R}_{i\bar{j}} = R_{i\bar{j}} + \mu(t) g_{i\bar{j}}.$$

Then

$$\begin{array}{rcl} \frac{\partial}{\partial t} \hat{R}_{i\bar{j}} & = & \frac{\partial}{\partial t} R_{i\bar{j}} + \mu(t) \frac{\partial}{\partial t} g_{i\bar{j}} + \mu' g_{i\bar{j}} \\ & = & \triangle_\varphi R_{i\bar{j}} + R_{i\bar{j}k\bar{l}} R_{\bar{k}l} - R_{i\bar{p}} R_{p\bar{j}} + \mu(t)(g_{i\bar{j}} - R_{i\bar{j}}) + \mu' g_{i\bar{j}} \\ & = & \triangle_\varphi \hat{R}_{i\bar{j}} + R_{i\bar{j}k\bar{l}} \hat{R}_{\bar{k}l} - \mu R_{i\bar{j}} - \hat{R}_{i\bar{p}} \hat{R}_{p\bar{j}} + 2\mu \hat{R}_{i\bar{j}} - \mu^2 g_{i\bar{j}} + \mu(\mu + 1) g_{i\bar{j}} - \mu \hat{R}_{i\bar{j}} + \mu' g_{i\bar{j}} \\ & = & \triangle_\varphi \hat{R}_{i\bar{j}} + R_{i\bar{j}k\bar{l}} \hat{R}_{\bar{k}l} - \mu \hat{R}_{i\bar{j}} - \hat{R}_{i\bar{p}} \hat{R}_{p\bar{j}} + 2\mu \hat{R}_{i\bar{j}} + \mu(\mu + 1) g_{i\bar{j}} - \mu \hat{R}_{i\bar{j}} + \mu' g_{i\bar{j}} \\ & = & \triangle_\varphi \hat{R}_{i\bar{j}} + R_{i\bar{j}k\bar{l}} \hat{R}_{\bar{k}l} - \hat{R}_{i\bar{p}} \hat{R}_{p\bar{j}} + (\mu(\mu + 1) + \mu') g_{i\bar{j}}. \end{array}$$

If we choose $\mu(0)$ such that $R_{i\bar{j}} + \mu g_{i\bar{j}} \geq 0$ at time $t = 0$ set $\mu(t)$ solves the following equation

$$\mu'(t) + \mu(\mu + 1) = 0$$

or

$$\mu(t) = \frac{C}{e^t - C}, \qquad \text{where } C = \frac{\mu(0)}{\mu(0) + 1}.$$

Then the evolution equation for $\hat{R}_{i\bar{j}}$ is:

$$\frac{\partial}{\partial t}\hat{R}_{i\bar{j}} = \triangle_{\varphi}\hat{R}_{i\bar{j}} + R_{i\bar{j}k\bar{l}}\hat{R}_{k\bar{l}} - \hat{R}_{i\bar{p}}\hat{R}_{p\bar{j}}.$$

Consequently, the non-negativity of $\hat{R}_{i\bar{j}}$ is preserved. This is because at the point where where $\hat{R}_{i\bar{j}}$ vanishes at least in one direction, we can show that the right hand side must be non-negative. In fact, set this direction as $\frac{\partial}{\partial z_1}$ and diagonalize the Ricci curvature at this point. Then

$$\frac{\partial}{\partial t}\hat{R}_{1\bar{1}} \geq R_{1\bar{1}j\bar{j}}\hat{R}_{j\bar{j}} - \hat{R}_{1\bar{1}}\hat{R}_{\bar{1}1} = \sum_{j=2}^{n} R_{1\bar{1}j\bar{j}}\hat{R}_{j\bar{j}} \geq 0.$$

□

## 2.3 On the holomorphic sectional curvature

Suppose $(M, g(0))$ is a Kähler manifold with positive orthogonal bisectional curvature. Roughly speaking, we want to prove that the lower bound of the holomorphic sectional curvature, if not positive, will be preserved or improved under appropriate other conditions. Now we give a proof to Theorem 1.4.

*Proof.* For any bisectional curvature type tensor $A_{i\bar{j}k\bar{l}}$, we define the operator $\Box$ as

$$\begin{aligned}\Box A_{i\bar{j}k\bar{l}} \quad = \quad & \triangle A_{i\bar{j}k\bar{l}} + A_{i\bar{j}p\bar{q}}A_{q\bar{p}k\bar{l}} - A_{i\bar{p}k\bar{q}}A_{p\bar{j}q\bar{l}} + A_{i\bar{l}p\bar{q}}A_{q\bar{p}k\bar{j}} \\ & + A_{i\bar{j}k\bar{l}} - \tfrac{1}{2}\left(R_{i\bar{p}}A_{p\bar{j}k\bar{l}} + R_{p\bar{j}}A_{i\bar{p}k\bar{l}} + R_{k\bar{p}}A_{i\bar{j}p\bar{l}} + R_{p\bar{l}}A_{i\bar{j}k\bar{p}}\right)\end{aligned}$$

Then the evolution equation for bisectional curvature is

$$\begin{aligned}\tfrac{\partial}{\partial t}R_{i\bar{j}k\bar{l}} \quad &= \quad \Box R_{i\bar{j}k\bar{l}} \\ &= \quad \triangle R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\bar{q}}R_{q\bar{p}k\bar{l}} - R_{i\bar{p}k\bar{q}}R_{p\bar{j}q\bar{l}} + R_{i\bar{l}p\bar{q}}R_{q\bar{p}k\bar{j}} \\ & \quad\quad + R_{i\bar{j}k\bar{l}} - \tfrac{1}{2}\left(R_{i\bar{p}}R_{p\bar{j}k\bar{l}} + R_{p\bar{j}}R_{i\bar{p}k\bar{l}} + R_{k\bar{p}}R_{i\bar{j}p\bar{l}} + R_{p\bar{l}}R_{i\bar{j}k\bar{p}}\right).\end{aligned}$$

Set

$$S_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \mu(t)(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{j\bar{k}}) = R_{i\bar{j}k\bar{l}} - \mu(g * g)_{i\bar{j}k\bar{l}}$$

and

$$S_{k\bar{l}} = R_{k\bar{l}} - \mu(n + 1)g_{k\bar{l}}.$$

At the point where $g_{i\bar{j}} = \delta_{i\bar{j}},$ we can re-write the evolution equation for $R$ as

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} &= \triangle\left(S_{i\bar{j}k\bar{l}} + \mu(g*g)_{i\bar{j}k\bar{l}}\right) + (S_{i\bar{j}p\bar{q}} + \mu(g*g)_{i\bar{j}p\bar{q}})(S_{q\bar{p}k\bar{l}} + \mu(g*g)_{q\bar{p}k\bar{l}}) \\
&\quad -(S_{i\bar{p}k\bar{q}} + \mu(g*g)_{i\bar{p}k\bar{q}})(S_{p\bar{j}q\bar{l}} + \mu(g*g)_{p\bar{j}q\bar{l}}) + S_{i\bar{j}k\bar{l}} + \mu(g*g)_{i\bar{j}k\bar{l}} \\
&\quad +(S_{i\bar{l}p\bar{q}} + \mu(g*g)_{i\bar{l}p\bar{q}})(S_{q\bar{p}k\bar{j}} + \mu(g*g)_{q\bar{p}k\bar{j}}) \\
&\quad -\frac{1}{2}\left(R_{i\bar{p}}S_{p\bar{j}k\bar{l}} + R_{p\bar{j}}S_{i\bar{p}k\bar{l}} + R_{k\bar{p}}S_{i\bar{j}p\bar{l}} + R_{p\bar{l}}S_{i\bar{j}k\bar{p}}\right) \\
&\quad -\frac{\mu}{2}\left(R_{i\bar{p}}(g*g)_{p\bar{j}k\bar{l}} + R_{p\bar{j}}(g*g)_{i\bar{p}k\bar{l}} + R_{k\bar{p}}(g*g)_{i\bar{j}p\bar{l}} + R_{p\bar{l}}(g*g)_{i\bar{j}k\bar{p}}\right) \\
&= \Box S_{i\bar{j}k\bar{l}} + 2\mu S_{i\bar{j}k\bar{l}} + \mu(S_{i\bar{j}}g_{k\bar{l}} + S_{k\bar{l}}g_{i\bar{j}}) + \mu^2(g*g)_{i\bar{j}p\bar{q}}(g*g)_{q\bar{p}k\bar{l}} \\
&\qquad -2\mu S_{i\bar{j}k\bar{l}} - \mu(S_{i\bar{l}k\bar{j}} + S_{i\bar{l}k\bar{j}}) - \mu^2(g*g)_{i\bar{p}k\bar{q}}(g*g)_{p\bar{j}q\bar{l}} \\
&\qquad +2\mu S_{i\bar{l}k\bar{j}} + \mu(S_{i\bar{l}}g_{k\bar{j}} + S_{k\bar{j}}g_{i\bar{l}}) + \mu^2(g*g)_{i\bar{l}p\bar{q}}(g*g)_{q\bar{p}k\bar{j}} \\
&\qquad +\mu(g*g)_{i\bar{j}k\bar{l}} - \mu\left(R_{i\bar{j}}g_{k\bar{l}} + R_{k\bar{l}}g_{i\bar{j}} + R_{i\bar{l}}g_{k\bar{j}} + R_{k\bar{j}}g_{i\bar{l}}\right) \\
&= \Box S_{i\bar{j}k\bar{l}} + \mu^2\left((n+2)g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}} - 2g_{i\bar{j}}g_{k\bar{l}} - 2g_{i\bar{l}}g_{k\bar{j}} + (n+2)g_{i\bar{l}}g_{k\bar{j}} + g_{i\bar{j}}g_{k\bar{l}}\right) \\
&\qquad +\mu(g*g)_{i\bar{j}k\bar{l}} + \mu\left((S_{i\bar{j}} - R_{i\bar{j}})g_{k\bar{l}} + (S_{k\bar{l}} - R_{k\bar{l}})g_{i\bar{j}} + (S_{i\bar{l}} - R_{i\bar{l}})g_{k\bar{j}} + (S_{k\bar{j}} - R_{k\bar{j}})g_{i\bar{l}}\right) \\
&= \Box S_{i\bar{j}k\bar{l}} + \mu\left((n+1)\mu + 1\right)(g*g)_{i\bar{j}k\bar{l}} \\
&\qquad +\mu\left((S_{i\bar{j}} - R_{i\bar{j}})g_{k\bar{l}} + (S_{k\bar{l}} - R_{k\bar{l}})g_{i\bar{j}} + (S_{i\bar{l}} - R_{i\bar{l}})g_{k\bar{j}} + (S_{k\bar{j}} - R_{k\bar{j}})g_{i\bar{l}}\right) \\
&= \Box S_{i\bar{j}k\bar{l}} + \mu\left((n+1)\mu + 1\right)(g*g)_{i\bar{j}k\bar{l}} \\
&\qquad -\mu(n+1)\mu\left(g_{i\bar{j}}g_{k\bar{l}} + g_{k\bar{l}}g_{i\bar{j}} + g_{i\bar{l}}g_{k\bar{j}} + g_{k\bar{j}}g_{i\bar{l}}\right) \\
&= \Box S_{i\bar{j}k\bar{l}} + \mu\left(1 - (n+1)\mu\right)(g*g)_{i\bar{j}k\bar{l}}
\end{aligned}
$$

Notice that $R_{i\bar{j}k\bar{l}} = S_{i\bar{j}k\bar{l}} + \mu(g*g)_{i\bar{j}k\bar{l}}$ and

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\mu(g*g)_{i\bar{j}k\bar{l}}\right) \\
= \;& \mu'(g*g)_{i\bar{j}k\bar{l}} + \mu\left(g_{i\bar{j}}(g_{k\bar{l}} - R_{k\bar{l}}) + g_{i\bar{l}}(g_{k\bar{j}} - R_{k\bar{j}}) + g_{k\bar{l}}(g_{i\bar{j}} - R_{i\bar{j}}) + g_{k\bar{j}}(g_{i\bar{l}} - R_{i\bar{l}})\right) \\
= \;& \mu'(g*g)_{i\bar{j}k\bar{l}} + 2\mu(1 - (n+1)\mu)(g*g)_{i\bar{j}k\bar{l}} - \mu(S_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{j}}S_{k\bar{l}} + S_{i\bar{l}}g_{k\bar{j}} + S_{k\bar{j}}g_{i\bar{l}})
\end{aligned}
$$

we have

$$
\begin{aligned}
(\frac{\partial}{\partial t} - \Box)S_{i\bar{j}k\bar{l}} &= \left(\mu((n+1)\mu - 1) - \mu'\right)(g*g)_{i\bar{j}k\bar{l}} \\
&\qquad +\mu\left(S_{i\bar{j}}g_{k\bar{l}} + S_{k\bar{l}}g_{i\bar{j}} + S_{i\bar{l}}g_{k\bar{j}} + S_{k\bar{j}}g_{i\bar{l}}\right) \\
&= \left(\mu((n+1)\mu - 1) - \mu'\right)(g*g)_{i\bar{j}k\bar{l}} \\
&\qquad +\mu\left(R_{i\bar{j}}g_{k\bar{l}} + R_{k\bar{l}}g_{i\bar{j}} + R_{i\bar{l}}g_{k\bar{j}} + R_{k\bar{j}}g_{i\bar{l}}\right) \\
&\quad -2\mu^2(n+1)(g*g)_{i\bar{j}k\bar{l}} \\
&= \left(\mu(-(n+1)\mu - 1) - \mu'\right)(g*g)_{i\bar{j}k\bar{l}} \\
&\qquad +\mu\left(R_{i\bar{j}}g_{k\bar{l}} + R_{k\bar{l}}g_{i\bar{j}} + R_{i\bar{l}}g_{k\bar{j}} + R_{k\bar{j}}g_{i\bar{l}}\right).
\end{aligned}
$$

Note that we assume that the positivity of the orthogonal bisectional curvature is preserved by the Kähler Ricci flow. Suppose $\mu(t) < 0$ is the lower bound of the evolved bisectional curvature such that $S_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \mu(t)(g*g)_{i\bar{j}k\bar{l}} \geq 0$. At the point and time where $S_{i\bar{j}k\bar{l}}$ reaches 0, then this minimum is realized by a holomorphic sectional curvature. Without loss of generality, we may assume that it is $S_{1\bar{1}1\bar{1}}(p, t_0) = 0$. Then at $t = t_0$,

$$R_{1\bar{1}1\bar{1}} = 2\mu.$$

We can diagonalize so that

$$R_{1\bar{1}k\bar{l}}(p, t) = \lambda_k \delta_{kl}, \forall\, k,\; l = 1, 2, \cdots n.$$

Here $\lambda_1 = 2\mu$ and

$$A = R_{1\bar{1}}(p,t) = \sum_{k=1}^{n} \lambda_k = +2\mu + \sum_{k=2}^{n} \lambda_k \geq c.$$

As in the Mok's argument [14], we have

$$\begin{aligned}
\frac{\partial}{\partial t} S_{1\bar{1}1\bar{1}} \mid_{(p,t)} &\geq S_{1\bar{1}k\bar{l}} S_{\bar{k}l\bar{1}1} + (\mu(-(n+1)\mu - 1) - \mu')\,(g * g)_{1\bar{1}1\bar{1}} + 2A\mu (g*g)_{1\bar{1}1\bar{1}} \\
&= \sum_{k=2}^{n} (\lambda_k - \mu)^2 + 2\,(\mu(-(n+1)\mu - 1) - \mu') + 4A\mu \\
&\geq \tfrac{1}{n-1} \left( \sum_{k=2}^{n} \lambda_k - (n-1)\mu \right)^2 + 2\,(\mu(-(n+1)\mu - 1) - \mu') + 4A\mu \\
&= \tfrac{1}{n-1} (A - (n+1)\mu)^2 + 2\,(\mu(-(n+1)\mu - 1) - \mu') + 4A\mu \\
&= \tfrac{A^2}{n-1} - 2\mu(1 - \tfrac{n-3}{n-1}A) + \tfrac{n+1}{n-1}(3-n)\mu^2 - 2\mu'.
\end{aligned}$$

From the last expression, it shows that for $n = 2, 3$, the lower bound of holomorphic sectional curvature is preserved (when the absolute lower bound is big enough). However, more should be true. In fact, when the minimum of $S_{i\bar{j}k\bar{l}}$ is achieved by holomorphic sectional curvature, by an argument of linear algebra (c.f. Lemma 2.1 and inequality 2.2 below. This appears explicitely in [2] (which eventually is due to R. Hamilton)), we can squiz a little more to obtain

$$\begin{aligned}
\frac{\partial}{\partial t} S_{1\bar{1}1\bar{1}} \mid_{(p,t)} &\geq 2S_{1\bar{1}k\bar{l}} S_{\bar{k}l\bar{1}1} + (\mu(-(n+1)\mu - 1) - \mu')\,(g * g)_{1\bar{1}1\bar{1}} + 2A\mu (g*g)_{1\bar{1}1\bar{1}} \\
&= 2\sum_{k=2}^{n} (\lambda_k - \mu)^2 + 2\,(\mu(-(n+1)\mu - 1) - \mu') + 4A\mu \\
&\geq \tfrac{2}{n-1} \left( \sum_{k=2}^{n} \lambda_k - (n-1)\mu \right)^2 + 2\,(\mu(-(n+1)\mu - 1) - \mu') + 4A\mu \\
&= \tfrac{2}{n-1} (A - (n+1)\mu)^2 + 2\,(\mu(-(n+1)\mu - 1) - \mu') + 4A\mu \\
&= \tfrac{2A^2}{n-1} - 2\mu(1 + \tfrac{4}{n-1}A) + \tfrac{4(n+1)}{n-1}\mu^2 - 2\mu' \\
&= \tfrac{2}{n-1}(A - 2\mu)^2 - \tfrac{8\mu^2}{n-1} - 2\mu + \tfrac{4(n+1)}{n-1}\mu^2 - 2\mu' \\
&= \tfrac{2}{n-1}(A - 2\mu)^2 + 4\mu^2 - 2\mu - 2\mu'.
\end{aligned}$$

Consequently, the lower bound of holomorphic sectional curvature is preserved and graduately improve to 0 as $t \to \infty$. No assumption on Ricci curvature needed to prove the preservation of lower bound here. However, if a positive lower bound of Ricci curvature is preserved, then the holomorphic sectional curvature shall becomes positive after finite time.

□

**Lemma 2.1.** *Let $A, B, C$ are complex matrix and $A, C$ are hermitian matrix. Suppose that the real quadritic form*

$$Q = A_{i\bar{j}} x^i x^{\bar{j}} + C_{i\bar{j}} y^k y^{\bar{l}} + 2Re(B_{ij} x^i y^j + B_{i\bar{j}} x^i y^{\bar{j}}) \geq 0, \qquad \forall\, x, y \in \mathbb{C}^n.$$

*Then, we have*

$$A_{i\bar{j}} C_{\bar{j}i} \geq |B_{ij}|^2 + |B_{i\bar{j}}|^2.$$

Now we indicate how we can apply this lemma to the preceding proof. Suppose that $S_{i\bar{j}k\bar{l}}$ achieve minimum in the direction $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_{\bar{1}}}$. Then, for any

$$x=\sum_{i=1}^{n} x^i \frac{\partial}{\partial z_i}, \qquad y=\sum_{i=1}^{n} y^i \frac{\partial}{\partial z_i}$$

we have

$$\frac{\partial^2}{\partial \epsilon^2} S(\frac{\partial}{\partial z_1}+\epsilon x, \frac{\partial}{\partial z_{\bar{1}}}+\epsilon \bar{x}, \frac{\partial}{\partial z_1}+\epsilon y, \frac{\partial}{\partial z_{\bar{1}}}+\epsilon \bar{y}) \mid_{\epsilon=0} \geq 0.$$

In other words, we have

$$S_{1\bar{1}k\bar{l}} y^k y^{\bar{l}} + S_{i\bar{j}1\bar{1}} x^i x^{\bar{j}} + 2Re(S_{1\bar{k}1\bar{j}} x^{\bar{k}} y^{\bar{j}} + S_{1\bar{k}j\bar{1}} x^{\bar{k}} y^j) \geq 0.$$

Applying the lemma above, we have

$$\begin{aligned} S_{1\bar{1}k\bar{l}} S_{\bar{k}l\bar{1}1} &\geq S_{1\bar{k}1\bar{j}} S_{\bar{1}k\bar{1}j} + S_{1\bar{k}j\bar{1}} S_{\bar{1}k\&bj1} && (2.1)\\ &\geq S_{1\bar{k}1\bar{j}} S_{\bar{1}k\bar{1}j} + S_{1\bar{1}j\bar{k}} S_{\bar{1}1\bar{j}k}. && (2.2) \end{aligned}$$

Now we give a proof to Theorem 1.5.

*Proof.* First we assume that holomorphic sectional curvature is still negative somewhere in $M$ and

$$R_{i\bar{j}} \geq \nu g_{i\bar{j}} > 0.$$

Following the notation of the previous proof in this section, we have (at the minimum of the holomorphic sectional curvature)

$$\begin{aligned} \tfrac{\partial}{\partial t} S_{1\bar{1}1\bar{1}} \mid_{(p,t)} &\geq 2S_{1\bar{1}k\bar{l}} S_{\bar{k}l\bar{1}1} + (\mu(-(n+1)\mu-1)-\mu')\,(g*g)_{1\bar{1}1\bar{1}} + 2A\mu(g*g)_{1\bar{1}1\bar{1}}\\ &= 2\sum_{k=2}^{n}(\lambda_k-\mu)^2 + 2\,(\mu(-(n+1)\mu-1)-\mu') + 4A\mu\\ &\geq \tfrac{2}{n-1}\left(\sum_{k=2}^{n}\lambda_k-(n-1)\mu\right)^2 + 2\,(\mu(-(n+1)\mu-1)-\mu') + 4A\mu\\ &= \tfrac{2}{n-1}(A-(n+1)\mu)^2 + 2\,(\mu(-(n+1)\mu-1)-\mu') + 4A\mu\\ &= \tfrac{2A^2}{n-1} - 2\mu(1+\tfrac{4}{n-1}A) + \tfrac{4(n+1)}{n-1}\mu^2 - 2\mu'\\ &= \tfrac{2}{n-1}(A-2\mu)^2 - \tfrac{8\mu^2}{n-1} - 2\mu + \tfrac{4(n+1)}{n-1}\mu^2 - 2\mu'\\ &= \tfrac{2}{n-1}(A-2\mu)^2 + 4\mu^2 - 2\mu - 2\mu'\\ &\geq \tfrac{2}{n-1}\nu^2 - 2\mu'. \end{aligned}$$

The last inequality hold since $\mu < 0$. This shows that at most finite time, the holomorphic sectional curvature will become positive everywhere.

Nowe we return to the case where the bisectional curvature is already positive. By assumption, we have

$$R_{i\bar{j}} \geq \nu g_{i\bar{j}}, \frac{1}{2} < \nu < 1,$$

and the bisectional curvature

$$R_{i\bar{j}k\bar{l}} \geq \mu(t)(g * g)_{i\bar{j}k\bar{l}}, \qquad \text{where } \mu(0) \geq 0.$$

Then,

$$\begin{aligned} & \tfrac{\partial}{\partial t}\left(\mu(g*g)_{i\bar{j}k\bar{l}}\right) \\ = \;& \mu'(g*g)_{i\bar{j}k\bar{l}} + \mu\left(g_{i\bar{j}}(g_{k\bar{l}} - R_{k\bar{l}}) + g_{i\bar{l}}(g_{k\bar{j}} - R_{k\bar{j}}) + g_{k\bar{l}}(g_{i\bar{j}} - R_{i\bar{j}}) + g_{k\bar{j}}(g_{i\bar{l}} - R_{i\bar{l}})\right) \\ \leq \;& \mu'(g*g)_{i\bar{j}k\bar{l}} + 2\mu(1-v)(g*g)_{i\bar{j}k\bar{l}}. \end{aligned}$$

Consequently, we have

$$(\tfrac{\partial}{\partial t} - \Box)S_{i\bar{j}k\bar{l}} \quad \geq \quad (\mu(2v - (n+1)\mu - 1) - \mu')\,(g*g)_{i\bar{j}k\bar{l}}.$$

Let

$$\mu(t) = \frac{Ce^{at}}{Ce^{at}+1}\cdot a, \qquad \text{where } C = \frac{\mu(0)}{1-\mu(0)} \text{ and } a = \frac{2\nu - 1}{n+1}.$$

Then

$$(\frac{\partial}{\partial t} - \Box)S_{i\bar{j}k\bar{l}} \geq 0.$$

□

# 3 Proof of Theorem 1.8

*Proof.* By Theorem 1.2, Ricci curvature is uniformly bounded from below over the entire flow. According to Perelman, the scalar curvature is uniformly bounded over the entire flow. Consequently, the Ricci curvature is uniformly bounded from below and above. Since orthogonal bisectional curvature is positive and the holomorphic sectional curvature bounded from below (Theorem 1.4), then all bisectional curvature is uniformly bounded over the entire Kähler Ricci flow. By Perelman's non-local collapsing lemma when curvature is bounded, we imply that the diameter is uniformly bounded from above. Consequently, the injectivity radius must have a uniform positive lower bound over the flow. Suppose that $J$ is the underlying complex structure of $M$. For any time sequence $\{t_i, i \in \mathbb{N}\}$, there exists a subsequence $\{g(t_i), i \in \mathbb{N}\}$ and a sequence of diffeomorphism $\rho_i : M \to M$ such that the sequence of pointed manifold $\{(M, \rho_i^* g(t_i), J_i = \rho_i^* J), i \in \mathbb{N}\}$ converges to a Kähler Ricci soliton $g_\infty$ with non-negative bisectional curvature in $M$, perhaps with a different complex structure $J_\infty$. Using Theorem 1.4, the bisectional curvature of the limit metric $g_\infty$ must be non-negative. Since $g_\infty$ is a Kähler Ricci soliton, then either the Ricci curvature of $g_\infty$ has no null direction at all or the Kähler manifold will be split into a product of at least two Kähler sub-manifolds. Then, $M$ must be decomposed as a product of two Kähler Submanifolds. Since we assume that $M$ is irreducible, this leads a contradiction. Thus, the Ricci curvature $R_{i\bar{j}}(g_\infty) > 0$ for any Kähler Ricci Soliton arised this way. Since $M$ is compact, then there

exists a small constant $\epsilon_0 > 0$ such that $R_{i\bar{j}}(g_\infty) > \epsilon_0$. It is then straightforward to see that there is a positive constant $\epsilon_0 > 0$ which doesn't depend on which time subsequences we choose. In other words, for $t \geq t_0$ big enough, the Ricci curvature is already positive; moreover, a positive uniform lower bound is a priori preserved over the Kähler Ricci flow after $t > t_0$. Following Theorem 1.5, the holomorphic sectional curvature must become positive after finite time beyond $t = t_0$. Since the orthogonal bisectional curvature is always positive before $t = \infty$, we show that the evolved Kähler metric $g(t)$, after at most finite time, must have positive bisectional curvature. Appealing to the Frankel conjecture, then $M$ is $\mathbb{CP}^n$. Then, the exponential convergence of the flow follows our earlier work [5] [6].

□

# 4 Proof of Theorem 1.10

First, let's state a parabolic version of Moser iteration lemma as in [7] without proof.

**Lemma 4.1.** *If the Poincare constant and the Sobolev constant of the evolving Kähler metrics $g(t)$ are both uniformly, and if a non-negative function $u$ satisfying the following inequality*

$$\frac{\partial}{\partial t} u \leq \triangle u + f(x,t)\, u, \qquad \forall a < t < b,$$

*where $|f|_{L^p(M,g(t))}$ is uniformly bounded by some constant $c$, then for any $\lambda \in (0,1)$ fixed, we have*

$$\max_{(1-\lambda)a+\lambda b \leq t \leq b} u \leq C(c, b-a, \lambda) \int_a^b \int_M u.$$

Now we prove Theorem 1.10.

*Proof.* We want to prove that each of the three conditions will implies the flow converges to a Fubni-study metrics exponentially. Naturally, this break into three cases.

**Case 1**. Let us first suppose that $E_1$ has a lower bound. Theorem 1.2 implies that $Ric(g(t)) > -1$ hold after finite time. The energy functional $E_1$ is monotonely decreases afterwards. Since $E_1$ is assumed to have a lower bound, then there exists a $t_0 \geq 0$ such that

$$\int_{t_0}^\infty d\,t \int_M |Ric(g(t)) - \omega(g(t))|^2\, \omega(g(t))^n \leq \epsilon(n).$$

According to Theorem 1.4 and the fact that the scalar curvature is uniformly bounded, the Riemannian curvature is uniformly bounded from below and above on the entire Kähler Ricci flow. Therefore, the diameter are uniformly bounded

on the Kähler Ricci flow as in [6]. Consequently, the injectivity radius must have a uniformly positive lower bounded over the entire flow. In other words, both Sobleve constant and Poincare constants of the evolving metrics are uniformly bounded from above. According to the iteration Lemma 4.1, the Ricci curvature is uniformly pinched towards identity by the constant $\epsilon(t) \to 0$ such that we have

$$1 - \epsilon(t) < Ric(g(t)) \leq 1 + \epsilon(t), \qquad \forall t > t_0.$$

In particular, let $\nu(t)$ to be the lower bound of $Ric(g(t))$. Then $\lim_{t\to\infty} \nu(t) = 1$. Appealing to Theorem 1.5, the lower bound of the bisectional curvature eventually improve to $\frac{1}{n+1}$. However, at any point $p \in M$, adopting an orthonomal frame, we have

$$\begin{aligned} R_{i\bar{i}}(p) &= \sum_{k=1}^{n} R_{i\bar{i}k\bar{k}} \\ &\geq \sum_{1\leq k\leq n; k\neq i} R_{i\bar{i}k\bar{k}} + R_{i\bar{i}i\bar{i}}. \end{aligned}$$

Thus, if the lower bound of the bisectional curvature improve to $\frac{1}{n+1}$ and the Ricci tensor approaches to the identity, then the full bisectional curvature approach to the bisectional curvature of a Fubini Study metric. Consequently, the underlying manifold is $\mathbb{CP}^n$. Following Theorem [6], the flow converges exponentially fast to a metric with constant bisectional curvature.

**Case 2**. Let us assume now that the Mabuchi energy has a uniform lower bound. Then, we have

$$\int_0^\infty d\,t \int_M |\nabla \frac{\partial \varphi}{\partial t}|^2_{\varphi(t)} \;\; \omega^n_{\varphi(t)} \leq C.$$

The evolution equation for $F(t) = |\nabla \frac{\partial\varphi}{\partial t}|^2_{\varphi(t)}$ is simple

$$\frac{\partial}{\partial t} F = \triangle_{\varphi(t)} F + F - \left(\frac{\partial\varphi}{\partial t}\right)_{\alpha\bar{\beta}} \cdot \left(\frac{\partial\varphi}{\partial t}\right)_{\bar{\alpha}\beta} - \left(\frac{\partial\varphi}{\partial t}\right)_{\alpha\beta} \cdot \left(\frac{\partial\varphi}{\partial t}\right)_{\bar{\alpha}\bar{\beta}}.$$

Following the iteration Lemma 4.1, we have

$$F = |\nabla \frac{\partial\varphi(t)}{\partial t}|^2_{\varphi(t)} \to 0.$$

Consider the evolution of $\int_M F$ over time.

$$\begin{aligned} \frac{d}{d\,t} \int_M F\, \omega^n_{\varphi(t)} &= \int_M F\, \omega^n_{\varphi(t)} + \int_M F(n-R)\omega^n_{\varphi(t)} \\ &\quad - \int_M \left(\frac{\partial\varphi}{\partial t}\right)_{\alpha\bar{\beta}} \cdot \left(\frac{\partial\varphi}{\partial t}\right)_{\bar{\alpha}\beta} \omega^n_{\varphi(t)} - \int_M \left(\frac{\partial\varphi}{\partial t}\right)_{\alpha\beta} \cdot \left(\frac{\partial\varphi}{\partial t}\right)_{\bar{\alpha}\bar{\beta}} \omega^n_{\varphi(t)}. \end{aligned}$$

Since the scalar curvature is uniformly bounded and $F \to 0$ pointwisely, we have

$$
\begin{aligned}
\lim_{A\to\infty}\int_A^{A+1} d\,t \int_M & |Ric(g(t)) - \omega(g(t))|^2\, \omega(g(t))^n \\
&= \lim_{A\to\infty}\int_A^{A+1} d\,t \int_M \left(\frac{\partial\varphi}{\partial t}\right)_{\alpha\bar\beta} \cdot \left(\frac{\partial\varphi}{\partial t}\right)_{\bar\alpha\beta} \omega_{\varphi(t)}^n \\
&= 0.
\end{aligned}
$$

Following the iteration Lemma again, we obtain

$$
1 - \epsilon(t) < Ric(g(t)) \leq 1 + \epsilon(t), \qquad \forall t > t_0.
$$

Appealing to Theorem 1.5, the bisectional curvature eventually pinch towards that of Fubini Study metric. Therefore the underlying manifold is $\mathbb{CP}^n$. Following Theorem [6], the flow converges exponentially fast to a metric with constant bisectional curvature.

**Case 3**. Let us assume that there exists a Kähler Einstein metric in the canonical Kähler class. By a well known theorem, the Mabuchi energy has a uniform lower bound in this class. We can reduce this to **Case 2** before and prove the exponential convergence of the Kähler Ricci flow.

We then complete our proof in all three cases. □

# 5 Future problems

The following questions are interesting:

1. Theorem 1.8 is an extension of Frankel conjecture. Does there exists a direct proof as in Siu-Yau, Morri's well known theorem?
2. In this paper, we numerate quite a few new Maximum principle theorem. Is the lower bound of Ricci curvature is always preserved in Kähler Ricci flow? Perleman's work seems to suggest that this is true.
3. It is known that the scalar curvature is bounded from above by exponential function over the Kähler Ricci flow. Does the same hold for bisectional curvature?
4. How to derive the bound of the scalar curvature without using Perelman's deep result?
5. In this paper, we use Perelman's result to derive scalar curvature bound. We obtain positive lower bound control on Ricci curvature after finite time via an argument of taking limit. Is that possible to have a more direct argument?
6. Is compact Kähler manifold with positive orthogonal bisectional curvature necessary have positive first Chern class (except dimension 1)? The answer should be yes, at least when dimension is high enough.

7. What happen in the case of positive 2 curvature operator in the sense of H. Chen [4].

8. Is there a (negative) lower bound of the holomophic sectional curvature which is preserved under the Kähler Ricci flow ? This lower bound shall depends on the initial Kähler metric. My inclination is that the answer shall be affirmative.

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